

HOMOGENEOUS MONGE-AMPÈRE EQUATIONS AND CANONICAL TUBULAR NEIGHBOURHOODS IN KÄHLER GEOMETRY

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ABSTRACT. We prove the existence of canonical tubular neighbourhoods around complex submanifolds of Kähler manifolds that are adapted to both the holomorphic and symplectic structure. This is done by solving the complex Homogeneous Monge-Ampère equation on the deformation to the normal cone of the submanifold. We use this to establish local regularity for global weak solutions, giving local smoothness to the (weak) geodesic rays in the space of (weak) Kähler potentials associated to a given complex submanifold. We also use it to get an optimal regularity result for naturally defined plurisubharmonic envelopes and for the boundaries of their associated equilibrium sets.

Dedicated to Findus on his birthday

1. INTRODUCTION

1.1. Canonical tubular neighbourhoods. Tubular neighbourhoods are used in differential and symplectic geometry to reveal the structure around submanifolds. As is well known, a submanifold of a complex manifold will in general not admit a tubular neighbourhood that is holomorphic, and with this in mind we investigate here the structure around a complex submanifold of a manifold that is Kähler.

To state our results, let $\pi: N_Y \rightarrow Y$ be the normal bundle of a complex submanifold Y of a complex manifold X and $\iota: Y \rightarrow N_Y$ be the inclusion of Y as the zero section. Note that N_Y is a holomorphic vector bundle that admits a holomorphic S^1 -action obtained by rotating the fibres. A smooth tubular neighbourhood of $Y \subset X$ is a diffeomorphism $T: U \rightarrow V \subset X$ between a neighbourhood U of $\iota(Y) \subset N_Y$ and a neighbourhood V of $Y \subset X$ such that

$$T \circ \iota = \text{id}_Y.$$

Theorem 1.1. *Suppose that Y is a complex submanifold of a Kähler manifold (X, ω) . Then there exists a smooth tubular neighbourhood*

$$T: U \rightarrow V \subset X$$

of $Y \subset X$ with the following properties:

- (1) U is S^1 -invariant and the pullback $T^*\omega$ is an S^1 -invariant Kähler form on U .
- (2) For any $u \in U$ the function $f_u: S^1 \rightarrow \mathbb{R}$, given by $f_u(e^{i\theta}) := T(e^{i\theta}u)$ extends to a holomorphic function $F_u: D \rightarrow \mathbb{C}$ from the unit disc $D \subset \mathbb{C}$ such that

$$F_u(0) = \pi(u) \text{ and } \left[DF_u|_0 \left(\frac{\partial}{\partial x} \right) \right] = u,$$

i.e. the holomorphic disc F_u is centered at $\pi(u)$ and points in the normal direction determined by u .

When Y is compact there is a canonical choice of T . In general, the germ of the tubular neighbourhood we construct is local and canonical, in the sense that at any point $\iota(p) \in \iota(Y)$ it depends only on the local structure of $(Y, \omega|_Y) \subset (X, \omega)$ around $p \in Y$.

Of course the map T need not be holomorphic, but the two properties above describe ways in which it interacts with the holomorphic structure (for example the statement that $T^*\omega$ is Kähler is not immediate given that T is not holomorphic). Even in cases where holomorphic tubular neighbourhoods do exist they will not in general have this property so what is produced above will be different. In fact the existence of such a tubular neighbourhood is highly non-trivial even when the submanifold is a single point in \mathbb{C} .

In the very special case that there exists a holomorphic S^1 -action on X that fixes Y pointwise inducing the usual action on N_Y obtained by rotating the fibers there is a natural holomorphic tubular neighbourhood of Y which encodes this symmetry. If in addition this action preserves ω then the germ of our T will agree with this holomorphic tubular neighbourhood. Since our construction is local this is also true locally around any point $p \in Y$ (see Proposition 4.7).

We point out that there is a standard way to produce tubular neighbourhoods that only satisfy the first property in the above theorem. In fact, for any choice of symplectic form $\tilde{\omega}$ on a neighbourhood U of $\iota(Y)$ such that $\iota^*\tilde{\omega} = \omega|_Y$ there exists, after shrinking U if necessary, a tubular neighbourhood $T: U \rightarrow X$ such that $T^*\omega = \tilde{\omega}$ (this follows by using an arbitrary tubular neighbourhood to pull back ω and then applying the relative version of Moser's Theorem [7, Theorem 7.4]). So by choosing such an $\tilde{\omega}$ that is Kähler and S^1 -invariant we get a tubular neighbourhood with property (1). This will of course depend on several choices, and from the point of view of Kähler geometry it is not clear why any particular choice is more natural than any other.

We are not aware of any standard construction that yields tubular neighbourhoods with property (2), but we suspect that they are relatively easy to produce. Thus the significance of the above theorem is that there exist tubular neighbourhoods with both properties simultaneously, which as we shall see can be made canonical.

1.2. HMAE on the deformation to the normal cone of Y . Really the main result of this paper is a proof of existence of regular solutions to a Dirichlet problem for a certain homogeneous Monge-Ampère equation (HMAE), and the tubular neighbourhood of Theorem 1.1 will then be constructed using the associated foliation.

Recall the *deformation to the normal cone* \mathcal{N}_Y of Y in X is the blowup

$$\pi: \mathcal{N}_Y \rightarrow X \times D$$

of $X \times D$ along $Y \times \{0\}$. The fiber of \mathcal{N}_Y over $0 \in D$ has two components, one which is isomorphic to the blowup $Bl_Y(X)$ of X along Y and the other being the exceptional divisor E which is isomorphic to the projective completion $\mathbb{P}(N_Y \oplus \mathbb{C})$ of the normal bundle N_Y . Letting $\mathcal{Y} \subset \mathcal{N}_Y$ denote the proper transform of $Y \times D$ we have that $\iota(Y) = E \cap \mathcal{Y}$. Similarly if $p \in Y$ we let \mathcal{D}_p denote the proper transform of $\{p\} \times D$, so $\iota(p) = E \cap \mathcal{D}_p$. The holomorphic S^1 -action on $X \times D$ given by

$$e^{i\theta} \cdot (x, \tau) = (x, e^{-i\theta} \tau) \tag{1}$$

lifts to \mathcal{N}_Y ; it preserves N_Y and induces the action on N_Y obtained by rotating the fibers.

Theorem 1.2. *There exists an S^1 -invariant neighbourhood V of \mathcal{Y} in \mathcal{N}_Y and a smooth closed S^1 -invariant real $(1, 1)$ -form Ω on V that gives a regular solution to the homogeneous Monge-Ampère equation with boundary data induced by ω . That is,*

- (1) $\Omega|_{V_\tau}$ is Kähler for all $\tau \in D$ where $V_\tau := \pi_D^{-1}(\tau) \cap V$,
- (2) $\Omega|_{V_\tau} = \omega|_{V_\tau}$ for all $\tau \in S^1$, and
- (3) $\Omega^{n+1} = 0$ on V .

The solution Ω is cohomologous to π_X^ω, i.e. there exists a (unique) smooth real valued S^1 -invariant function Φ on V which is zero on V_τ for $\tau \in S^1$ and such that*

$$\Omega = \pi_X^*\omega + dd^c\Phi.$$

When Y is compact there is a canonical choice of (V, Ω) . In general the germ around \mathcal{Y} of any such S^1 -invariant regular solution is unique.

We observe that since the central fibre of \mathcal{N}_Y is of a different topological type to the general fibre, there can never be a regular solution defined on all of \mathcal{N}_Y . In this sense a local result such as Theorem 1.2 is the best one could hope for. The uniqueness part of Theorem 1.2 is also novel since it does not involve any hypothesis on the behaviour of the solutions near the boundary of V .

We explain briefly how such an Ω gives rise to a tubular neighbourhood. Since the work of Bedford-Kalka [3] it has been known that regular solutions to the HMAE generate associated *Monge-Ampère foliations* by holomorphic curves as follows: the kernel of the form Ω defines an integrable distribution of complex lines in the tangent bundle of V . By Frobenius' theorem it induces a foliation by holomorphic curves, which because of property (1) is transverse to the fibers V_τ . In the case under consideration, Ω is S^1 -invariant and hence so is the foliation (see Figure 1).

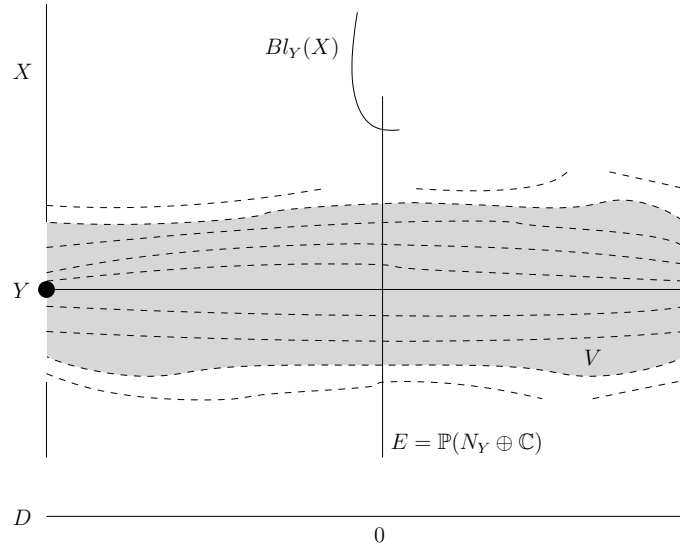


FIGURE 1. Local regularity of HMAE in neighbourhood of the proper transform of $Y \times D$. Dotted lines represent the leaves of the foliation.

We shall say that a leaf of the foliation is *complete* if it covers the base D (i.e. the restriction to this leaf of the projection to D is a surjection onto D). From the S^1 -invariance it follows that $\Omega|_{\mathcal{D}_p} = 0$ for all $p \in Y$, thus each \mathcal{D}_p is a complete leaf of the foliation. By continuity we can then find an S^1 -invariant neighbourhood of \mathcal{Y} consisting solely of complete leaves that pass through $N_Y \subset E$, so after shrinking V we can assume that V is foliated by such leaves. Note that $U := V_0 = V \cap \pi^{-1}(0)$ is an S^1 -invariant neighbourhood of $\iota(Y)$ in N_Y . By flowing along the leaves of this foliation we get for each $\tau \in D$ a diffeomorphism $T_\tau : U \rightarrow V_\tau$ which remarkably is a symplectomorphism, i.e.

$$T_\tau^*(\Omega|_{V_\tau}) = \Omega|_{V_0}. \quad (2)$$

The map $T_1 : U \rightarrow V_1$ is the desired tubular neighbourhood, and thanks to (2) it has property (1) of Theorem 1.1, while the existence of the holomorphic leaves assures that it has property (2) (the details are provided in Section 4).

Definition 1.3. We say that a solution (V, Ω) to the HMAE is *complete* if V is foliated by complete leaves of the Monge-Ampère foliation, $V_0 \subseteq N_Y$ and whenever $u \in V_0$ then $\tau u \in V_0$ for all $\tau \in D$.

Given a regular solution (V, Ω) as in Theorem 1.2 we get a complete regular solution by shrinking V if necessary.

Let ζ denote the vector field which generates the S^1 -action on V . From the fact that Ω is cohomologous to $\pi_X^* \omega$ it follows that the S^1 -action is Hamiltonian, in the sense described in the next theorem.

Theorem 1.4. *Let (V, Ω) be a regular solution to the HMAE as in Theorem 1.2. Then the function $H := L_{J\zeta} \Phi$ is a Hamiltonian for the S^1 -action, in that it satisfies*

$$dH = \iota_\zeta \Omega.$$

Moreover H is constant along the leaves of the Monge-Ampère foliation associated to Ω . If (V, Ω) is complete then $H \geq 0$ with equality precisely on \mathcal{Y} .

At least when Y is compact we can use this Hamiltonian to select a canonical regular solution (V_{can}, Ω_{can}) to the HMAE.

Definition 1.5. Given a complete solution (V, Ω) with Hamiltonian H we define the radius $rad(V, \Omega)$ of (V, Ω) to be the supremum of all $\lambda \geq 0$ such that $\partial H^{-1}([0, \lambda)) \subseteq V$. We then define the *canonical radius* Λ_{can} to be the supremum of $rad(V, \Omega)$ over all complete solutions (V, Ω) .

Note that the radius of a complete solution could be zero, and hence the canonical radius Λ_{can} could also be zero. But at least when Y is compact we clearly have that $rad(V, \Omega) > 0$, and thus $\Lambda_{can} > 0$.

Theorem 1.6. *Assume that Y is compact (or more generally $\Lambda_{can} > 0$). Then there exists a unique complete solution (V_{can}, Ω_{can}) to the HMAE as in Theorem 1.2 with*

$$rad(V_{can}, \Omega_{can}) = \Lambda_{can}$$

and

$$H_{can} < \Lambda_{can}.$$

This canonical solution is maximal in the following sense. If (V, Ω) is any other complete solution to the HMAE as in Theorem 1.2 then for any $\lambda < rad(V, \Omega)$ we have that

$$H^{-1}([0, \lambda)) = H_{can}^{-1}([0, \lambda))$$

and there $\Omega = \Omega_{can}$.

Thus when $\Lambda_{can} > 0$ the function $H_{can}|_{V_1}$ gives a canonical smooth function defined on a neighbourhood of Y in X (that depends on the Kähler structure of X near Y), whose level sets give a canonical "flow" away from Y , which we will revisit below.

1.3. Local regularity of weak solutions. For topological reasons one cannot find global regular solutions of the HMAE on $\mathcal{N}_{\mathcal{Y}}$ but instead of looking for local regular solutions one can consider global weak solutions. Using a variant of the Perron envelope, for any fixed number c one can construct a closed positive $(1, 1)$ -current Ω_w on $\mathcal{N}_{\mathcal{Y}}$ cohomologous to $\pi_X^* \omega - c[E]$ which restricts to ω on X_τ for $\tau \in S^1$, and which is maximal in the sense of pluripotential theory. Moreover when X is compact and c is small

$$\Omega_w^{n+1} = 0$$

in the sense of Bedford-Taylor, thus Ω_w is a globally defined weak solution to the HMAE.

Theorem 1.7. *Suppose Y is compact (or more generally $\Lambda_{can} > 0$). Then Ω_w is equal to Ω_{can} on $H_{can}^{-1}([0, c))$. In particular Ω_w is smooth and regular on a neighbourhood of \mathcal{Y} .*

In particular this gives a local regularity result for the weak geodesic rays that are naturally associated to the deformation to the normal cone (see Section 1.6).

1.4. Optimal regularity of plurisubharmonic envelopes. As an application of the above ideas we shall give a new regularity result for some naturally defined envelopes that occur in pluripotential theory. Let Y be a compact complex submanifold of a (not necessarily compact) Kähler manifold (X, ω) and let $\lambda > 0$ be a parameter. We consider the envelope

$$\psi_\lambda := \sup\{\psi \in PSH(X, \omega) : \psi \leq 0 \text{ and } \nu_Y(\psi) \geq \lambda\}$$

where $PSH(X, \omega)$ denotes the set of ω -plurisubharmonic functions (i.e. upper semicontinuous L^1_{loc} functions ψ such that $\omega + dd^c\psi$ is a positive current) and $\nu_Y(\psi)$ denotes the Lelong-number of ψ along Y . Given this data, the *equilibrium set* is defined to be

$$S_\lambda := \psi_\lambda^{-1}(0)$$

whose complement S_λ^c is a neighbourhood of Y . We introduce the following definition of regularity that captures what we shall prove:

Definition 1.8. We say that ψ_λ has *optimal regularity* if

- (1) S_λ is smoothly bounded.
- (2) On $S_\lambda^c \setminus Y$ the envelope ψ_λ is smooth and also

$$(\omega + dd^c\psi_\lambda)^{n-1} \neq 0 \quad \text{on } S_\lambda^c \setminus Y.$$

- (3) There is a family $\{D_q\}$ of holomorphic discs in S_λ^c , parametrized by points q in the exceptional divisor in $Bl_Y X$, such that the restriction of $\omega + dd^c\psi_\lambda$ to each D_q is zero. Moreover each D_q passes through Y , the boundary of D_q lies in ∂S_λ and the family $\{D_q\}$ foliates the blowup of S_λ^c along Y (see Figure 3).

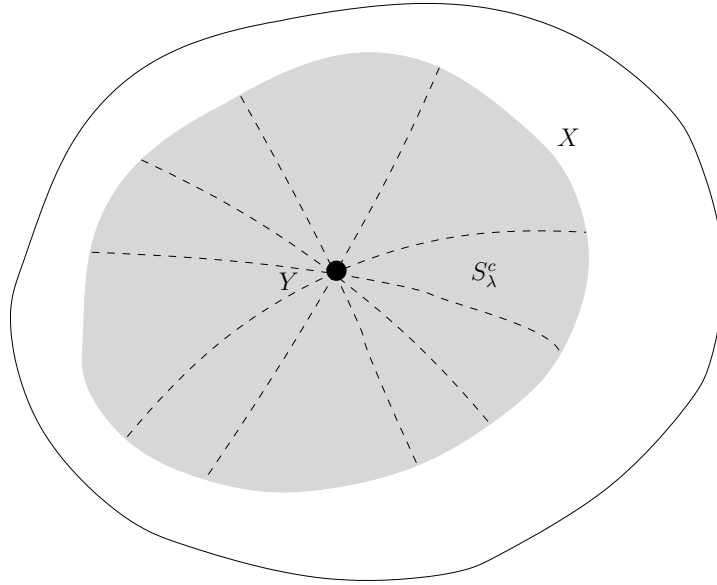


FIGURE 2. Optimal Regularity around a submanifold Y . Dotted lines represent the discs D_q .

Recall that Λ_{can} denotes the canonical radius.

Theorem 1.9. Assume that Y is compact (or more generally $\Lambda_{can} > 0$). Then for λ small enough (i.e. $\lambda < \Lambda_{can}$) the envelope ψ_λ has optimal regularity. Moreover the corresponding holomorphic discs D_p all have area λ , and the boundaries ∂S_λ vary smoothly with λ .

In fact $\partial S_\lambda = H_{can}^{-1}(\lambda) \cap V_1$ where H_{can} is the Hamiltonian function of the canonical solution (V_{can}, Ω_{can}) to the HMAE associated to the data $Y \subset (X, \omega)$.

The above result is interesting even in the simplest case that Y is a point in an open set in \mathbb{C}^n . This is then a purely local statement and we shall use the optimal regularity to prove the existence of a flow in \mathbb{C}^n by sets with a certain reproducing property:

Theorem 1.10. *Let ϕ be a smooth strictly plurisubharmonic function on the unit ball B in \mathbb{C}^n . Define*

$$\psi_\lambda := \sup\{\psi \in PSH(B) : \psi \leq \phi - \lambda \ln |z|^2\}$$

and

$$S_\lambda := \{\psi_\lambda + \lambda \ln |z|^2 = \phi\}.$$

Then for small λ (i.e. $\lambda < \Lambda_{can}$) the envelope ψ_λ has optimal regularity. Also for any bounded holomorphic function f on $B_\lambda := B \setminus S_\lambda$ we have

$$\frac{1}{\text{vol}(B_\lambda)} \int_{B_\lambda} f \frac{(dd^c \phi)^n}{n!} = f(0). \quad (3)$$

Here Λ_{can} is the canonical radius of the HMAE associated to the data $\{0\} \subset (B, dd^c \phi)$.

The reproducing property in (3) arises in the theory of complex moments and the planar Hele-Shaw flow in fluid mechanics (see Section 1.6) and gives a generalization of this flow to all complex dimensions.

1.5. Outline of Proofs. The key tool that we will use to prove the existence of local regular solutions to the HMAE is an openness theorem of Donaldson [12]. This states that on a product $X \times D$, where X is a compact Kähler manifold, the set of boundary conditions for which there exists a regular solution to the HMAE is open (in a suitable topology). Recall that such a boundary condition consists of a family of cohomologous Kähler forms ω_τ on X for $\tau \in S^1$. If ω_τ were independent of τ then the trivial solution to the HMAE obtained by pulling back ω to the product is regular. Thus we have regular solutions to any small perturbation of such a boundary condition.

Of course we are not interested in the product, but in the deformation to the normal cone \mathcal{N}_Y . By working locally around a fixed point $p \in Y$ it is possible by a simple change of coordinates to translate the HMAE on a neighbourhood of \mathcal{D}_p to one on a product. The cost in this change is that whereas we originally were seeking a solution with S^1 -invariant boundary data, the new boundary data will no longer be S^1 -invariant. We shall show, however, that by shrinking attention to a sufficiently small neighbourhood of p any boundary data can be made sufficiently close to an invariant one; and thus we can apply Donaldson's Openness Theorem. Thus for any point $p \in Y$ we get a local regular solution to the HMAE around \mathcal{D}_p .

We then observe that regular solutions to HMAE enjoy some strong uniqueness properties (even if they are defined locally) coming from the existence of the associated foliation. We use this to show that these local solutions glue together to give a regular solution in a neighbourhood of the proper transform \mathcal{Y} of $Y \times D$. The fact that the solutions we produce are cohomologous to $\pi_X^* \omega$ gives the existence of a Hamiltonian function, and we use this to define the notion of canonical radius Λ_{can} and to get a canonical regular solution to the HMAE.

To prove that, if Y is compact (or more generally if $\Lambda_{can} > 0$), the weak solution agrees with the canonical regular one in a neighbourhood of \mathcal{Y} we use the Legendre transform of the potential Φ to deduce that on some neighbourhood of \mathcal{Y} this Φ is bounded from above by a local potential for the weak solution. The fact that $dd^c \Phi$ is harmonic along the leaves of the Monge-Ampère foliation allows us to use the maximum principle to deduce the reverse inequality.

For the result about the envelopes ψ_λ , the key point is that these can be shown to agree with the Legendre transform of the potential for the canonical regular solution. The associated foliation on \mathcal{N}_Y can then be translated to this optimal regularity result on X .

1.6. Comparison with previous works. Tubular Neighbourhoods: It is well known that holomorphic tubular neighbourhoods around a complex submanifold $Y \subset X$ need not exist. In fact if one does exist then the exact sequence $0 \rightarrow TY \rightarrow TX|_Y \rightarrow N_Y \rightarrow 0$ must split holomorphically. There are stronger results on non-existence, for instance it is a theorem of Van de Ven [39] that the only connected submanifolds of projective space that admit holomorphic tubular neighbourhoods are linear subspaces.

The local structure around a holomorphic submanifold Y is studied in seminal works of Grauert [14] and Griffiths [15], in which some basic notions in algebraic geometry are developed, for example infinitesimal neighbourhoods, notions of positivity of vector bundles and subvarieties, and some vanishing theorems (see [8] for a survey). In this, and related works, what is often sought after is a “transverse foliation” by which is meant a family of disjoint subvarieties S_y for $y \in Y$ such that $S_y \cap Y = \{y\}$. We have avoided the use of this terminology, but what we produce is clearly analogous only we do not require that these subvarieties vary holomorphically in y . This highlights the key difference: the existence of such a foliation varying holomorphically places strong restrictions on the normal bundle, but by allowing a weaker notion we have existence always. Of course one of the main uses of the deformation to the normal cone in algebraic geometry is to get around the non-existence of holomorphic tubular neighbourhoods (as used, for example, in intersection theory). Thus it is not particularly surprising that this same deformation appears here.

Geodesics in the space of Kähler Metrics: The Dirichlet problem for the complex HMAE has a long history, going back at least as far as the fundamental work of Bedford-Taylor [2]. The existence of smooth (or regular) solutions is a difficult and much studied problem and can depend in a subtle way on the boundary data (see [16] for a survey). Following work of Semmes [35], Mabuchi [23], Donaldson [12], it is known that in the case of compact fibres the existence of solutions can be interpreted as geodesic rays in the space of Kähler metrics, and the existence of regular solutions has deep implications to the theory of extremal metrics (for example [9, 30, 24, 25, 26, 33, 34, 38] among others). In the above we consider this problem for the deformation to the normal cone of a submanifold Y . This family is the simplest non-trivial example of a *test-configuration* that lies at the heart of the Yau-Tian-Donaldson conjecture connecting the existence of a constant scalar curvature Kähler metric with the algebro-geometric notion of K-stability, and has been studied from a number of points of view (for example [1, 19, 28, 29, 40]). So another interpretation of Theorem 1.7 is that the Phong-Sturm geodesic in the space of Kähler metrics associated to the test-configuration given by the deformation to the normal cone of a submanifold Y is regular near the orbit of Y .

Envelopes, Equilibrium Set and the Hele Shaw flow: The kind of envelopes we consider above are basic objects in the study of the Monge-Ampère equation (see for example [21, Ch. 6]). They are related to the asymptotic behaviour of the partial Bergman kernel involving holomorphic sections that vanish to a particular order along the submanifold Y . This was first studied in detail in the toric case by Schiffman-Zelditch [36] who introduced the name *forbidden region* essentially for the complement of what we call the equilibrium set. This was then taken up by Berman [5] in the general projective case (from whom we have taken the terminology *equilibrium set*) who proves, among other things, that the envelopes are $C^{1,1}$, and then again by Berman-Demailly [6] in the setting of big Kähler classes. We refer the reader also to Ross-Singer [27] and Ross-Witt-Nyström [31] where these envelopes are considered further.

In the particular case that Y is a single point in a Riemann surface (Σ, ω) , the tubular neighbourhoods produced here describe a model of the Hele-Shaw flow with empty initial condition and permeability encoded by ω . This has been studied by Hedenmalm-Shimorin [17] for real-analytic hyperbolic Riemann surfaces. Using the Hele-Shaw flow they construct an exponential map $H\text{Sexp}$ which has the property that the image of any concentric circle intersects orthogonally the image of any ray emanating from zero. One can check that this implies that their exponential map differs from ours, and does not share the properties listed in Theorem 1.1. The Hele-Shaw flow on a general Riemann surface is the topic of previous work of the authors [32] in which we prove the $n = 1$ case of Theorem 1.10. The link is made through the fact that the Hele-Shaw flow is characterized by its complex moments, or said another way the reproducing property (3). In this way we can consider Theorem 1.10 as a generalization of the Hele-Shaw flow to a point in \mathbb{C}^n (but do not claim that this has any fluids interpretation).

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2. PRELIMINARIES

2.1. Notation and Terminology. Deformation to the normal cone: Let Y be a complex submanifold of a complex manifold X (which we always take to be connected, closed as a subset and not equal to X). The *deformation to the normal cone* \mathcal{N}_Y of Y is the blowup

$$\mathcal{N}_Y \xrightarrow{\pi} X \times D$$

of $X \times D$ along $Y \times \{0\}$. The *exceptional divisor* $E = \pi^{-1}(Y \times \{0\})$ is isomorphic to the projective completion $\mathbb{P}(N_Y \oplus \mathbb{C})$ of the normal bundle N_Y of Y . The map π is an isomorphism away from E and the *proper transform* of a submanifold $Z \subset X \times D$ not contained in $Y \times \{0\}$ is the closure of $\pi^{-1}(Z \setminus Y \times \{0\})$ in \mathcal{N}_Y . We also let π_X and π_D denote the projections to X and D respectively.

Plurisubharmonic functions: The following notions can be found in [11, 21]. For an open subset U of a complex manifold X we denote by $PSH(U)$ the space of plurisubharmonic functions $\phi: U \rightarrow \mathbb{R} \cup \{-\infty\}$ on U . We say a function ϕ on U is *pluriharmonic* if both ϕ and $-\phi$ are plurisubharmonic, and ϕ is *strictly plurisubharmonic* if its curvature $dd^c\phi$ is a strictly positive current. Any Kähler form ω can be written locally as $dd^c\phi$ for some smooth strictly plurisubharmonic function ϕ , which is uniquely defined up to addition by pluriharmonic functions. Given a closed real $(1, 1)$ -form α on X we let $PSH(\alpha)$ be the set of upper-semicontinuous L^1_{loc} functions ψ such that $\alpha + dd^c\psi$ is a positive current. So if $\alpha = dd^c u$ on U then $\psi \in PSH(\alpha)$ if and only if $u + \psi \in PSH(U)$.

Foliations: A complex k -dimensional foliation on a n -dimensional complex manifold X consists of a set \mathcal{F} of k -dimensional complex submanifolds that cover X , that are pairwise disjoint, and so that any $x \in X$ is contained in a neighbourhood U for which there is a smooth mapping $f: U \rightarrow \mathbb{R}^{2n-2k}$ of maximal rank such that $\{\mathcal{L} \cap U : \mathcal{L} \in \mathcal{F}\} = \bigcup_{c \in \mathbb{R}^{2n-2k}} f^{-1}(c)$. We call the members of \mathcal{F} the *leaves* of the foliation. By the Frobenius Theorem, any involutive k -dimensional distribution of the tangent bundle of X is integrable, and thus this distribution is in fact the tangent space to the leaves of a uniquely defined foliation. Moreover if this distribution is complex (i.e. consists of complex subspaces of the tangent bundle) then the leaves of this foliation are complex submanifolds by the Theorem of Levi-Civita (see the Appendix of [21]).

C^2 -norms: When f is a smooth function on an open set $U \subseteq \mathbb{C}^n$ we will let the C^2 -norm of f be defined as

$$\|f\|_{C^2(U)} := \sup\{|f(z)| : z \in U\} + \sup\{\max(|f_{x_1}|, |f_{y_1}|, \dots, |f_{x_n}|, |f_{y_n}|) : z \in U\} + \sup\{\max(|f_{x_1 x_1}|, |f_{x_1 y_1}|, \dots, |f_{y_n y_n}|) : z \in U\}.$$

On a complex manifold X we choose an open cover of coordinate patches U_i together with a partition of unity $\sum_i \chi_i$, and the C^2 -norm of a smooth function f on X will be computed as

$$\sum_i \|\chi_i f\|_{C^2(U_i)}.$$

The C^2 -norm of functions defined on $X \times S^1$ is then defined in the obvious way.

2.2. The Dirichlet problem for the Homogeneous Monge-Ampère Equation. Suppose that $\pi : V \rightarrow D$ is a surjective map from a complex $(n+1)$ dimensional manifold V with boundary to the closed unit disc D such that all fibres $V_\tau := \pi^{-1}(\tau)$ are manifolds.

In this paper V will always be either a product $V = X \times D$ or a subset of the deformation to the normal cone \mathcal{N}_Y of a submanifold $Y \subset X$.

Definition 2.1. We shall refer to a smooth family of Kähler forms ω_τ on V_τ for $\tau \in S^1$ as *boundary data* for V . If V is a subset of \mathcal{N}_Y then the Kähler form ω on X induces the boundary condition $\omega_\tau = \omega|_{V_\tau}$ which we refer to as the *boundary condition induced by ω* .

Definition 2.2. We say that a smooth closed real $(1, 1)$ -form Ω on V is a *regular solution to the HMAE* with boundary data ω_τ if

- (1) $\Omega|_{V_\tau}$ is Kähler for all $\tau \in D$,
- (2) $\Omega|_{V_\tau} = \omega_\tau$ for all $\tau \in S^1$, and
- (3) $\Omega^{n+1} = 0$ on V .

The distribution defined by the kernel of a regular solution Ω to the HMAE is integrable (since Ω is closed), complex (since Ω is $(1, 1)$), and one-dimensional and transverse to the fibers (since Ω is nondegenerate along the fibers) (see [3] or [12]). Thus by the Frobenius integrability theorem there is 1-dimensional complex foliation of V , and by construction the restriction of Ω to each such leaf vanishes. We shall refer to this as the *Monge-Ampère foliation* determined by Ω .

2.3. Donaldson's Openness Theorem.

Theorem 2.3 (Donaldson's Openness Theorem). *Let (X, ω) be compact Kähler and set $V = X \times D$. Suppose that for some function $F(z, \tau)$ on V the form $\Omega_F = \pi_X^* \omega + dd^c F$ is a regular solution to the HMAE with boundary data $\omega_\tau := \omega + dd^c F(\cdot, \tau)$. Then any smooth function $g = g(z, \tau)$ on $X \times S^1$ that is sufficiently close to $F|_{X \times S^1}$ with respect to the C^2 -norm can be uniquely extended to smooth function G on V such that $\Omega_G = \pi_X^* \omega + dd^c G$ is a regular solution to the HMAE with boundary data $\omega + dd^c g(\cdot, \tau)$.*

Clearly this applies when $F \equiv 0$, and in fact we will only use the above theorem in this form (i.e. to obtain perturbations of this trivial solution). For convenience we record here the precise consequence that we need in the non-compact case.

Theorem 2.4. *There is an $\epsilon > 0$ such that the following holds: if $\phi_\tau, \tau \in S^1$ is a smooth family of smooth strictly plurisubharmonic functions on the unit ball $B_1 \subset \mathbb{C}^n$ such that*

- (i) *for all $\tau \in S^1$ the function ϕ_τ is equal to $|z|^2$ near ∂B_1 ,*
- (ii) $\|\phi_\tau - |z|^2\|_{C^2(B_1 \times S^1)} < \epsilon,$

then there exists a unique smooth plurisubharmonic function Φ on $B_1 \times D$ with $\Phi(z, \tau) = \phi_\tau(z)$ for any $\tau \in S^1$, Φ being equal to $|z|^2$ near $\partial B_1 \times D$, and such that $\Omega := dd^c \Phi$ is a regular solution to the HMAE on $B_1 \times D$ with boundary data $\omega_\tau = dd^c \phi_\tau$.

Moreover if ρ is a holomorphic S^1 -action on $B_1 \times D$ which covers the usual rotation on the disc (or its inverse), leaving the boundary data ϕ_τ and the function $|z|^2$ fixed, then the solution Φ is also invariant under this action.

One approach to prove this is simply to observe that Donaldson's proof of Theorem 2.3 works in this noncompact setting because we require ϕ_τ to be equal to $|z|^2$ near the boundary. But instead of relying on this fact, we shall show how this non-compact version can actually be deduced from the compact case.

Proof. Let $X = \mathbb{P}^n$, and consider the inclusions $B_1 \subset \mathbb{C}^n \subset X$. Then the Kähler form $dd^c|z|^2$ on B_1 can clearly be extended to a Kähler form ω on \mathbb{P}^n . As mentioned above, the trivial solution to the HMAE on $X \times D$ with boundary data induced by ω is regular.

Now let ϕ_τ be a family of smooth plurisubharmonic functions that are equal to $|z|^2$ near the boundary of B_1 . The function $g(z, \tau) := \phi_\tau(z) - |z|^2$ then extends by zero to $\mathbb{P}^n \times S^1$. By Donaldson's Openness Theorem there exists an $\epsilon > 0$ such that if the C^2 -norm of g is less than ϵ then g extends to a smooth function G on $\mathbb{P}^n \times D$ such that $\Omega := \pi_{\mathbb{P}^n}^* \omega + dd^c G$ is a regular solution to the HMAE. We can now set $\Phi = G + |z|^2$.

The fact that G is zero near $\partial B_1 \times D$ relies on aspects of Donaldson's proof of Theorem 2.3. The first step in that proof is a construction of a fibre bundle $W \rightarrow X$ a Kähler manifold (that is locally modelled on the cotangent bundle) so that there is a correspondence between Kähler forms of the form $\omega + dd^c F$ and the images of certain sections of W (the requirement being that this image is a "LS-manifold" in Donaldson's terminology). Thus a boundary data ω_τ defines a family of LS-submanifolds of W parameterized by $\tau \in S^1$. In our case the LS-submanifold is locally the graph of $\partial \phi_\tau$. The second step is to show that the foliation defined by a regular solution to the HMAE with this boundary data lifts to a family of holomorphic discs in W parameterized by X whose boundary lie in these LS-submanifolds. Moreover any holomorphic disc with this boundary condition that is sufficiently close to this family is actually a leaf of this foliation [12, Prop 2]. We now let G_t be the regular solution with boundary values $t g$, $t \in [0, 1]$. The LS-submanifolds at time t will thus be locally defined as $\partial((1-t)|z|^2 + t\phi_\tau)$. Also note at time 0 we have the trivial solution, and thus foliation at that time is given by the constant discs $\{z\} \times D$. Since these LS submanifolds do not change with t over a neighbourhood of ∂B_1 it follows that the constant discs for z near ∂B_1 remain to be part of the foliation, which forces G to be zero near $\partial B_1 \times D$.

Any solution Φ' with the stated properties will extend to a global solution on \mathbb{P}^n which by the uniqueness part of Theorem 2.3 is equal to $G + |z|^2$, thus showing that $\Phi = \Phi'$. That Φ is invariant under a holomorphic action which fixes the boundary data and $|z|^2$ now follows from the uniqueness of Φ , since any rotation of Φ gives a regular solution to the HMAE with the same boundary values. \square

Lemma 2.5. *If Ω is a regular solution to the HMAE on $B_1 \times D$ which is invariant under the action ρ of S^1 where*

$$\rho(e^{i\theta})(z', z'', \tau) := (e^{i\theta} z', z'', e^{-i\theta} \tau) = (e^{i\theta} z_1, \dots, e^{i\theta} z_r, z_{r+1}, \dots, z_n, e^{-i\theta} \tau)$$

then the constant discs $\{(0, z'')\} \times D$ are all leaves of the associated Monge-Ampère foliation.

Proof. We know that the Monge-Ampère foliation is ρ -invariant so since any point of the form $(0, z'', 0)$ is fixed by ρ the leaf passing through that point must be ρ -invariant itself. We let $f_i(\tau)$, $1 \leq i \leq r$ be the i :th coordinate of the point of that leaf in the fiber over $\tau \in D$, then f is holomorphic at least in some small neighbourhood of the origin. From the

invariance follows that for any $e^{i\theta} \in S^1$, $f(e^{i\theta}\tau) = e^{-i\theta}f(\tau)$ which immediately implies that f must be constantly zero (e.g. from looking at the winding number). Thus since the leaf is locally constant it must be globally constant, and hence equal to the constant disc $\{(0, z'')\} \times D$. \square

3. LOCAL REGULAR SOLUTIONS

3.1. Local Existence. Let Y be a complex submanifold of a Kähler manifold (X, ω) and consider the family $\mathcal{N}_Y \xrightarrow{\pi} D$ given by the deformation to the normal cone of Y . We recall that we say that a leaf of a one dimensional foliation on a subset of \mathcal{N}_Y is *complete* if the projection from this leaf to the base D is a surjection.

Definition 3.1. A *local regular solution* to the HMAE with boundary data induced by ω for a subset $A \subset Y$ consists of a pair (V, Φ) such that

- (1) V is an open S^1 -invariant neighbourhood of the proper transform of $A \times D$ in \mathcal{N}_Y such that all fibers V_τ are smooth manifolds,
- (2) Φ is a smooth real valued S^1 -invariant function on V which is zero on V_τ , $\tau \in S^1$,
- (3) $\Omega := \pi_X^* \omega + dd^c \Phi$ is a regular solution to the HMAE on V ,
- (4) V is the union of complete leaves of the Monge-Ampère foliation determined by Ω , and if a leaf intersects \mathcal{Y} then it must be equal to \mathcal{D}_p for some $p \in Y$.

Note that it follows from the definition that if (V, Φ) is a local regular solution to the HMAE for $A \subseteq Y$ then \mathcal{D}_p is part of the Monge-Ampère foliation for each $p \in A$.

Proposition 3.2. *Let $p \in Y$. Then there exists a local regular solution (V, Φ) to the HMAE for the subset $\{p\}$.*

As discussed previously, the idea is to reduce the problem to finding regular solutions to the HMAE over a neighbourhood of \mathcal{D}_p which looks like a ball times the disc, i.e. the product case, but with a twisted boundary condition. To set this up, let z_1, \dots, z_n be coordinates centered at p chosen so that Y is given locally by $z_1 = \dots = z_r = 0$ (these coordinates will later be chosen with additional properties). For simplicity we shall write $z = (z', z'')$ where $z' = (z_1, \dots, z_r)$ and $z'' = (z_{r+1}, \dots, z_n)$. Let $f : B_\delta \rightarrow X$ denote the corresponding holomorphic embedding of the ball of radius δ , where δ is chosen small enough so that Y is given by $z_1 = \dots = z_r = 0$ on the chart $f^{-1} : U \rightarrow B_\delta$. Let also $\omega_f := f^* \omega$.

Consider the map $\Gamma : B_\delta \times D^\times \rightarrow U \times D^\times$ given by

$$\Gamma(z, w) = (f(wz', z''), w).$$

From the definition of the blowup one easily sees that this map extends to a biholomorphism from $B_\delta \times D$ to an open set $V \subset \mathcal{N}_Y$ that contains \mathcal{D}_p . Henceforth we shall let

$$\Gamma : B_\delta \times D \rightarrow V \tag{4}$$

denote this extended biholomorphism. Thus Γ^{-1} is a holomorphic chart on \mathcal{N}_Y containing \mathcal{D}_p .

Recall the holomorphic action ρ of S^1 on $B_\delta \times D$:

$$\rho(e^{i\theta}) \cdot (z', z'', \tau) := (e^{i\theta} z', z'', e^{-i\theta} \tau) = (e^{i\theta} z_1, \dots, e^{i\theta} z_r, z_{r+1}, \dots, z_n, e^{-i\theta} \tau).$$

Under the biholomorphism Γ the action ρ is the same as the S^1 action on \mathcal{N}_Y considered above restricted to $V = \Gamma(B_\delta \times D)$. We shall therefore refer to this data as a ρ -chart. By abuse of notation we will let the holomorphic action of S^1 on B_1 given by

$$e^{i\theta} \cdot (z', z'') := (e^{i\theta} z', z'')$$

also be denoted by ρ .

Proposition 3.3. *There exists a local regular solution to the HMAE for the subset $\{p\}$ if and only if there exists a ρ -chart $f^{-1} : U \rightarrow B_\delta$ centered at p together with a smooth real ρ -invariant function Φ' on $B_\delta \times D$ such that $\Omega' := dd^c \Phi'$ is a regular solution to the HMAE with boundary condition $\omega_\tau := \rho(\tau)^* \omega_f$.*

Proof. If (V, Φ) is a local regular solution to the HMAE for the subset $\{p\}$ then pick a ρ -chart $f^{-1} : U \rightarrow B_\delta$ centered at p such that $\Gamma(B_\delta \times D) \subseteq V$. Let $\Phi'(z, \tau) := \phi \circ \rho(\tau)(z) + \Phi \circ \Gamma(z, \tau)$, where $dd^c \phi = \omega_f$. Since the usual S^1 -action on \mathcal{N}_Y restricts to ρ on the coordinate chart $B_\delta \times D$, Φ' is ρ -invariant, and since Γ is a biholomorphism Ω' is a regular solution to the HMAE on $B_\delta \times D$. One easily checks that it has boundary values $\omega_\tau := \rho(\tau)^* \omega_f$.

For the other direction, let us assume that we have a ρ -chart and an ρ -invariant function Φ' on $B_\delta \times D$ such that $\Omega' := dd^c \Phi'$ is a regular solution to the HMAE with boundary condition $\omega_\tau := \rho(\tau)^* \omega_f$. Let $\phi := \Phi'|_{B_\delta \times \{1\}}$. Thanks to Lemma 2.5 we know that $\{0\} \times D$ is part of the Monge-Ampère foliation on $B_\delta \times D$, so by continuity there exists a neighbourhood V' of $\{0\} \times D$ consisting of complete leaves. Since the foliation is ρ -invariant we can take the union of all $\rho(e^{i\theta})V'$ which is then a ρ -invariant neighbourhood V' of $\{0\} \times D$ consisting of complete leaves, and we now call this neighbourhood V' . Because Γ is a biholomorphism, $\Gamma^* \omega = \rho(\tau)^* \omega_f$ on $B_\delta \times S^1$ and ρ corresponds to the restriction of the usual S^1 -action on \mathcal{N}_Y to $V = \Gamma(V')$ we get that (V, Φ) where $\Phi := \Phi' \circ \Gamma^{-1} - \phi \circ f^{-1} \circ \pi_X$ is a local regular solution to the HMAE for the subset $\{p\}$. \square

Thus it is enough to find a ρ -chart $f^{-1} : U \rightarrow B_\delta$ centered at p together with a ρ -invariant function Φ' such that $\Omega' := dd^c \Phi'$ is a regular solution to the HMAE on $B_\delta \times D$ with boundary data $\rho(\tau)^* \omega_f$. The first observation is that, given a ρ -chart, if ω_f happened to be ρ -invariant then the boundary data is independent of τ , and so we have the trivial regular solution $\Phi'(z, \tau) := \phi(z)$, where ϕ is ρ -invariant and $dd^c \phi = \omega_f$. Of course ω_f will not in general be ρ -invariant, but we shall show that one always can find charts such that ω_f is approximately ρ -invariant. Thus Donaldson's Openness Theorem can be applied to give the desired local regular solution.

Lemma 3.4. *There exists a ρ -chart centered at p such that $\omega_f = dd^c \phi$ with*

$$\phi(z) = |z|^2 + O(|z|^3).$$

Proof. For completeness we include the standard argument. Let $u = (u', u'')$ with $u' = (u_1, \dots, u_r)$ and $u'' = (u_{r+1}, \dots, u_n)$ be coordinates centered at p defining some ρ -chart, and thus Y is given locally by $u' = 0$. Also choose a potential ψ for ω_f . The fact that ψ is real valued implies that the Taylor expansion around 0 takes the form

$$\psi(u) = \alpha + \operatorname{Re} \left(\sum_i \alpha_i u_i + \beta_i u_i^2 \right) + \sum_{ij} \gamma_{ij} u_i \bar{u}_j + o(|u|^2)$$

for some real coefficients $\alpha, \alpha_i, \beta_i$ with

$$\gamma_{ij} = \frac{\partial^2 \psi}{\partial u_i \partial \bar{u}_j} \Big|_{u=0}.$$

Clearly $h(u) := \alpha + \operatorname{Re}(\sum_i \alpha_i u_i + \beta_i u_i^2)$ is pluriharmonic since it is the real part of a holomorphic function.

As ψ is strictly plurisubharmonic, Γ is a positive definite hermitian. Thus by choosing an orthonormal basis with respect to the hermitian form given by Γ one gets a linear change of coordinates P so that $P^* \Gamma P = I$. Moreover this can be achieved whilst preserving a given subspace (say by first picking such an orthonormal basis for this subspace). Thus there exists a matrix with block form

$$P := \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

so that $P^*\Gamma P = I$. Letting $z := Pu$ we get a new ρ -chart f' and

$$\psi(z) = h(z) + |z|^2 + O(|z|^3).$$

Now $\omega_{f'} = dd^c\phi$ where $\phi(z) := \psi(z) - h(z)$ which clearly is of the right form. \square

A chart as in Lemma 3.4 will be called an admissible ρ -chart. We see that in such a chart the local potential ϕ is close in C^2 to $|z|^2$ near 0. The idea is now to find a smooth strictly plurisubharmonic function $\tilde{\phi}$ on B_1 that is equal to ϕ near 0 while (after a scaling and a constant shift) being ϵ -close in C^2 to $|z|^2$ on the whole of B_1 and actually equal to $|z|^2$ near ∂B_1 . This will subsequently allow us to apply the local version of Donaldson's Openness Theorem (Theorem 2.4) to get a local regular solution for $\{p\}$.

To get $\tilde{\phi}$ we will glue together ϕ with $(1+a)|z|^2 - 2b$ where $0 < b \ll a \ll 1$ and for this we need the following lemma:

Lemma 3.5. *Let $\alpha(z)$ and $\beta(z)$ be smooth strictly plurisubharmonic functions on some domain $U \subseteq \mathbb{C}^n$ containing the origin, and assume that for some $b > 0$ we have that $\alpha(0) > \beta(0) + b$ while $\alpha < \beta - b$ on ∂U . Then there exists a smooth strictly plurisubharmonic function $u(z)$ on U which is equal to α near 0, equal to β near ∂U , and such that*

$$\|u - \beta\|_{C^2(U)} \leq b + \|\alpha - \beta\|_{C^2(U)} + \frac{1}{b} \|d(\alpha - \beta)\|_{C^0(U)}^2, \quad (5)$$

where $\|d\gamma\|_{C^0(U)}$ denotes the maximum of $\|\frac{\partial\gamma}{\partial x_i}\|_{C^0(U)}$ and $\|\frac{\partial\gamma}{\partial y_i}\|_{C^0(U)}$ for $1 \leq i \leq n$.

Proof. Note that $\max(\alpha, \beta)$ is strictly plurisubharmonic, equal to α near 0 and equal to β near ∂U , but not smooth. Instead we let f be a smooth even non-negative bump function on the real line with unit integral, support in $(-b, b)$ which is bounded from above by $1/b$, and then define $u(z)$ to be

$$u(z) := \int_{\mathbb{R}} \max\{\alpha(z) + \lambda, \beta(z)\} f(\lambda) d\lambda.$$

That u is smooth, strictly plurisubharmonic and has the right behaviour near 0 and ∂U is easily checked. Note that

$$u(z) - \beta(z) = (\alpha(z) - \beta(z)) \int_{\beta(z) - \alpha(z)}^{\infty} f(\lambda) d\lambda + \int_{\beta(z) - \alpha(z)}^{\infty} \lambda f(\lambda) d\lambda,$$

and the bound of the C^2 -norm of $u - \beta$ on U follows from simply differentiating this expression and using the obvious bounds. \square

Proposition 3.6. *Given an admissible ρ -chart $f^{-1} : U \rightarrow B_\delta$ centered at p and an $\epsilon > 0$ we can find a smooth strictly plurisubharmonic function $\tilde{\phi}$ on B_1 such that for some constant c , $\omega_f = cdd^c\tilde{\phi}$ near 0, while at the same time $\tilde{\phi} = |z|^2$ near ∂B_1 and*

$$\|\tilde{\phi} - |z|^2\|_{C^2(B_1)} < \epsilon.$$

Proof. Let ϕ be a local potential on B_δ such that $dd^c\phi = \omega_f$ and $\phi(z) = |z|^2 + O(|z|^3)$. We are going to glue ϕ with $\beta := (1+a)|z|^2 - 2b$ where $0 < b \ll a$ are some small numbers yet to be determined. Clearly $\phi(0) > \beta(0) + b$. Since

$$|z|^2 = (1+a)|z|^2 - 3b$$

when $|z|^2 = 3b/a$ we get that for b small enough $\phi < \beta - b$ on ∂B_σ where say $\sigma = 3\sqrt{b/a}$. For this to make sense we need to choose b small enough so that $\sigma < \delta$ in order for ϕ to be defined on B_σ . By Lemma 3.5 there is a smooth strictly plurisubharmonic function u on B_σ which is equal to ϕ near 0, equal to β near ∂B_σ and with

$$\|u - \beta\|_{C^2(B_\sigma)} \leq b + \|\phi - \beta\|_{C^2(B_\sigma)} + \frac{1}{b} \|d\phi - d\beta\|_{C^0(B_\sigma)}^2.$$

We note that as b and thus σ tends to 0, $\|\phi - \beta\|_{C^2(B_\sigma)}$ will converge to $\|a|z|^2 - 2b\|_{C^2(\{0\})} = 2(a+b)$. Also for small b we have that

$$\|d\phi - d\beta\|_{C^0(B_\sigma)} \approx 2a\sigma = 6\sqrt{ab}.$$

Put together this means that by letting b be small enough we can get the C^2 -norm of u to approach $38a$. So then we pick a such that $38a < \epsilon/2$, and we will be able to get a smooth strictly plurisubharmonic function u defined on some small ball B_σ which is equal to ϕ near 0, equal to β near ∂B_σ and with

$$\|u - \beta\|_{C^2(B_\sigma)} < \epsilon.$$

Now let $\tilde{\phi}$ be defined as $(1+a)^{-1}(u+b)$ on B_σ , and as $\tilde{\phi}$ now is equal to $|z|^2$ near ∂B_σ we can extend it to B_1 by simply letting it be $|z|^2$ on $B_1 \setminus B_\sigma$. It is now immediate that this $\tilde{\phi}$ has the desired properties. \square

We are now ready to prove our local existence result.

Proof of Proposition 3.2. Pick an admissible ρ -chart $f^{-1} : U \rightarrow B_\delta$ centered at p , let $\epsilon > 0$ be that from Theorem 2.4, and let $\tilde{\phi}$ be a function on B_1 with the properties stated in Proposition 3.6. Thus

$$\|\tilde{\phi} - |z|^2\|_{C^2(B_1)} < \epsilon.$$

Let

$$\tilde{\phi}_\tau(z) := \tilde{\phi}(\tau z', z'') = \tilde{\phi} \circ \rho(\tau)(z).$$

As is easily seen

$$\|\tilde{\phi}_\tau - |z|^2\|_{C^2(B_1 \times S^1)} = \|\tilde{\phi} - |z|^2\|_{C^2(B_1)},$$

so it follows from Theorem 2.4 that there exists a ρ -invariant function Φ' such that $\Phi'(z, \tau) = \tilde{\phi}_\tau(z)$ when $\tau \in S^1$ and such that $\Omega' := dd^c \Phi'$ is a regular solution to the HMAE on $B_1 \times D$.

Now on some small ball $B_\sigma \subseteq B_\delta$ we have that $\omega_f = cdd^c \tilde{\phi}$ and thus on $B_\sigma \times S^1$ we have that $c\omega_\tau = \rho(\tau)^* \omega_f$. It follows that $c\Phi'$ restricted to $B_\sigma \times D$ is a ρ -invariant regular solution to HMAE with the right boundary values. Proposition 3.3 now tells us that we get a local regular solution to the HMAE for $\{p\}$. \square

3.2. Patching. We now show how the local solutions provided in Section 3.1 patch together. This will follow from a local uniqueness property for these regular solutions.

Proposition 3.7. *If (V^α, Φ^α) is a local regular solution to the HMAE for a subset $A \subseteq Y$ and (V^β, Φ^β) is a local regular solution to the HMAE for a subset $B \subseteq Y$ then in fact $\Phi^\alpha = \Phi^\beta$ in some neighbourhood of the proper transform of $(A \cap B) \times D$.*

Proof. Pick a point $p \in A \cap B$ and a ρ -chart $f^{-1} : U \rightarrow B_\delta$ centered at p such that $\Gamma(B_\delta \times D) \subseteq (V^\alpha \cap V^\beta)$. Let ϕ be a function such that $dd^c \phi = \omega_f$ and let $\Phi' := \phi \circ \rho(\tau)(z) + \Phi^\alpha \circ \Gamma(z, \tau)$ and $\Phi'' := \phi \circ \rho(\tau)(z) + \Phi^\beta \circ \Gamma(z, \tau)$. Then Φ' and Φ'' are two ρ -invariant regular solutions to the HMAE on $B_\delta \times D$ with the same boundary values. Let \mathcal{L} be a complete leaf of the foliation associated to Ω' , clearly we have that Φ' is harmonic along \mathcal{L} . Since $\Phi' = \Phi''$ on the boundary of \mathcal{L} while Φ'' is subharmonic on \mathcal{L} it follows from the maximum principle that $\Phi' \geq \Phi''$ on \mathcal{L} . We know that the constant disc $\{0\} \times D = \Gamma^{-1}(\mathcal{D}_p)$ is part of the Monge-Ampère foliation of $\Omega' := dd^c \Phi'$ (as well as that of $\Omega'' := dd^c \Phi''$) and thus a neighbourhood of $\{0\} \times D$ is foliated by such complete leaves, so in this neighbourhood $\Phi' \geq \Phi''$. But arguing the same way using complete leaves of the foliation associated to Ω'' now gives us that in fact $\Phi' = \Phi''$ in a neighbourhood of $\{0\} \times D$. Pulling back by Γ^{-1} yields the identity of Φ^α and Φ^β near \mathcal{D}_p , and we are done. \square

Remark 3.8. For spaces $\pi_D : V \rightarrow D$ where the fibers are compact manifolds without boundary there exist general uniqueness results for the Homogeneous Monge-Ampère equation even for weak solutions. However the local uniqueness result stated here is quite different in that it does not depend on the behaviour of the solution near the boundary of V .

If (V, Φ) is a local regular solution to the HMAE and $C \subseteq V_1$ then we let V_C be defined as the union of all leaves of the associated Monge-Ampère foliation which connect to $C \times \{1\} \subseteq \mathcal{N}_Y$. Note that this is an S^1 -invariant subset of \mathcal{N}_Y . Given $\epsilon > 0$ we will let C^ϵ denote the set of points in X within an ϵ distance of C (measured using the Kähler metric).

Lemma 3.9. *If A, B are two compact subsets of Y and (V^α, Φ^α) is a local regular solution to the HMAE for A while (V^β, Φ^β) is a local regular solution to the HMAE for B , then there exists a local regular solution (V, Φ) to the HMAE for $A \cup B$. Moreover if C is a compact subset of X such that $C \cap Y = A$ then we can choose (V, Φ) so that for some $\epsilon > 0$ we have that $V_{C^\epsilon}^\alpha \subseteq V$ and $\Phi = \Phi^\alpha$ on $V_{C^\epsilon}^\alpha$.*

Proof. By Proposition 3.7, for any point $p \in A \cap B$ we can pick an S^1 -invariant neighbourhood U_p of D_p in $V^\alpha \cap V^\beta$ where $\Phi^\alpha = \Phi^\beta$, consisting only of complete leaves of the associated foliation. Let U denote the union of those neighbourhoods. Now we pick a compact subset $C \subseteq V_1^\alpha$ such that $C \cap Y = A$. By property (4) of a local regular solution (see Definition 3.1) we have that $V_{C^{2\epsilon}}^\alpha \cap \mathcal{Y}$ is equal to the proper transform of $(C^{2\epsilon} \cap Y) \times D$. This implies that when ϵ is small enough

$$V_{C^\epsilon}^\alpha \cup U \cup (V_\beta \setminus V_{C^{2\epsilon}}^\alpha)$$

is a neighbourhood of the proper transform of $(A \cup B) \times D$. We then define Φ to be equal to Φ^α on $V_{C^\epsilon}^\alpha$, while letting it be Φ^β on $(V_\beta \setminus V_{C^{2\epsilon}}^\alpha)$, and then equal to either one on U . This now gives local regular solution (V, Φ) to the HMAE for $A \cup B$. \square

The following gives a proof of Theorem 1.2. As before the data consists of a Kähler manifold (X, ω) (not necessarily compact) together with a complex submanifold $Y \subset X$.

Theorem 3.10. *There exists a local regular solution (V, Φ) to the HMAE for Y . In particular it means that we have an S^1 -invariant neighbourhood V of \mathcal{Y} in \mathcal{N}_Y and a smooth closed S^1 -invariant real $(1, 1)$ -form $\Omega := \pi_X^* \omega + dd^c \Phi$ on V that gives a regular solution to the homogeneous Monge-Ampère equation with boundary data induced by ω , i.e.*

- (1) $\Omega|_{V_\tau}$ is Kähler for all $\tau \in D$,
- (2) $\Omega|_{V_\tau} = \omega|_{V_\tau}$ for all $\tau \in S^1$, and
- (3) $\Omega^{n+1} = 0$ on V .

Moreover the germ around \mathcal{Y} of any such S^1 -invariant regular solution is unique.

Proof. Using Proposition 3.2 there is a collection $(V^{\alpha_i}, \Phi^{\alpha_i})$, ($i \in I$ with $I = \{1, \dots, N\}$ or $I = \mathbb{N}$) of local regular solutions to the HMAE for compact subsets $A_i \subseteq Y$ such that $\{A_i^\circ\}_{i \in I}$ is a locally finite cover of Y . Let $C_1 := A_1$. Then by Lemma 3.9, for some $\epsilon_1 > 0$ we can find a locally regular solution (V^2, Φ^2) to the HMAE for $(A_1^{\epsilon_1} \cap Y) \cup A_2$ such that $V_{A_1^{\epsilon_1}}^{\alpha_1} \subseteq V^2$ and so that $\Phi^2 = \Phi^{\alpha_1}$ there. For $k \geq 2$ we let $C_k := C_{k-1}^{\epsilon_{k-1}} \cup A_k$ and do the same thing, and we will get a sequence of local regular solutions (V^k, Φ^k) to the HMAE for $\bigcup_{i=1}^k A_i$. We note that the subsets $V_{C_k}^k$ are all S^1 -invariant and increase with k , and for $l > k$, $\Phi^l = \Phi^k$ on $V_{C_k}^k$. Let $V := \bigcup_{k=1}^\infty V_{C_k}^k$ and define Φ by letting it be Φ^k on $V_{C_k}^k$, then it is immediate that (V, Φ) has the properties described in the theorem. This proves the existence part, while the uniqueness of the germ follows directly from Proposition 3.7. \square

In fact we can say a bit more than just that the germ of any regular solution is unique, but for this we need to introduce the notion of complete regular solutions.

Definition 3.11. We will call a pair (V, Ω) a *complete regular solution* to the HMAE (or in short a *complete solution*) if it is a regular solution to the HMAE as in Theorem 1.2 (so in particular it is cohomologous to $\pi_X^*\omega$) such that V is foliated by complete leaves of the Monge-Ampère foliation, $V_0 \subseteq N_Y$ and whenever $u \in V_0$ then $\tau u \in V_0$ for all $\tau \in D$.

Given a regular solution (V, Ω) as in Theorem 1.2, by shrinking V we can always get a nontrivial complete solution.

Proposition 3.12. *If (V, Ω) and (V', Ω') are two complete solutions and $V_0 \subseteq V'_0$ then $V \subseteq V'$ and $\Omega = \Omega'$ on V .*

Proof. Let Φ and Φ' be the potentials of Ω and Ω' respectively. Pick a point $u \in V_0$ and let \mathcal{L}_t denote the leaf in the Monge-Ampère foliation of Ω that passes through tu for $t \in [0, 1]$. Let also \mathcal{L}'_t denote the leaf in the Monge-Ampère foliation of Ω' that passes through tu . We know that $\mathcal{L}_0 = \mathcal{L}'_0$. Let $T := \sup\{t : \mathcal{L}_t = \mathcal{L}'_t\}$. We claim that $T = 1$. Indeed, if $\mathcal{L}_t = \mathcal{L}'_t$ then there is a neighbourhood $U \subseteq V \cap V'$ of \mathcal{L}_t consisting of complete leaves of the Monge-Ampère foliation of Ω . On any such leaf $\Phi' - \Phi$ is subharmonic, and since this function is zero on the boundary of the leaf we get that $\Phi \geq \Phi'$. Similarly, using a neighbourhood of $\mathcal{L}_t = \mathcal{L}'_t$ consisting of leaves of the foliation associated to Ω' we get that in fact $\Phi = \Phi'$ in a neighbourhood of \mathcal{L}_t , and hence the foliations agree there. This implies that the set of points $t \in [0, 1]$ such that $\mathcal{L}_t = \mathcal{L}'_t$ is open. On the other hand it is closed since the leaves vary continuously with t . This thus shows that $\mathcal{L}_1 = \mathcal{L}'_1$ and also that $\Phi = \Phi'$ on that leaf. Since V is foliated by such leaves it gives us the proposition. \square

Remark 3.13. Note that in the proof we only used the facts that V and V' were foliated by complete leaves, $V_0 \subseteq N_Y$, V_0 connected and $V_0 \subseteq V'_0$.

Let ζ denote the vector field generated by the S^1 -action on V . From the fact that Ω is cohomologous to $\pi_X^*\omega$ follows that the S^1 -action is Hamiltonian, in the sense described in the next theorem.

Theorem 3.14. *Let (V, Ω) be a regular solution to the HMAE as in Theorem 1.2. Then the function $H := L_{J\zeta}\Phi$ is a Hamiltonian for the S^1 -action, in that it satisfies*

$$dH = \iota_\zeta \Omega. \quad (6)$$

H is constant along the leaves of the Monge-Ampère foliation associated to Ω . Furthermore if (V, Ω) is complete then $H \geq 0$ with equality precisely on \mathcal{Y} .

Proof. From the definition of the dd^c -operator we get that

$$dL_{J\zeta}\Phi = \iota_\zeta dd^c\Phi = \iota_\zeta(\Omega - \pi_X^*\omega),$$

but on the other hand clearly $\iota_\zeta \pi_X^*\omega = 0$, and thus $H := L_{J\zeta}\Phi$ is a Hamiltonian for the S^1 -action.

The Lie derivative of H with respect to the vector field ξ obtained by the flow along a leaf of the associated foliation is

$$L_\xi H = \iota_\xi dH = 0$$

where we have used (6) and that by definition ξ lies in the kernel of Ω . Thus H is constant along leaves. As Φ is zero on \mathcal{Y} and ζ is tangential to \mathcal{Y} we get that $H = 0$ along \mathcal{Y} .

It remains to prove that H is strictly positive away from \mathcal{Y} when the solution is complete. Since H is constant along the leaves of the foliation it suffices to consider H on the discs $\{\tau u : u \in V_0 \setminus \iota(Y), \tau \in D\} \subseteq V_0$. We have that

$$H(u) = \frac{d}{dt}_{t=0} \Phi(e^{t/2}u)$$

and since $\pi_X^* \omega$ restricts to zero on the disc the function $t \mapsto \Phi(e^t u)$ is strictly convex and 0 at $t = -\infty$, which implies that $H(u) > 0$ as long as $u \notin \iota(Y)$. \square

3.3. Canonical solutions. We will now show how to define a canonical complete solution (V_{can}, Ω_{can}) , at least when Y is compact.

Definition 3.15. Given a complete solution (V, Ω) with Hamiltonian H we define the radius $rad(V, \Omega)$ of (V, Ω) to be the supremum of all $\lambda \geq 0$ such that $\partial H^{-1}([0, \lambda)) \subseteq V$. We then define the *canonical radius* Λ_{can} to be the supremum of $rad(V, \Omega)$ over all complete solutions (V, Ω) .

Note that the radius of a complete solution could be zero, and hence the canonical radius Λ_{can} could also be zero. But at least when Y is compact we clearly have that $rad(V, \Omega) > 0$, and thus $\Lambda_{can} > 0$.

Lemma 3.16. *If (V, Ω) is a complete solution and $\lambda > 0$ then $(H^{-1}([0, \lambda)), \Omega)$ is a new complete solution.*

Proof. Since H is constant along the leaves we see that $H^{-1}([0, \lambda))$ is still foliated by complete leaves, and from the proof of Theorem 3.14 we see that $H(\tau u) \leq H(u)$ when $\tau \in D$, showing that the discs $\{\tau u : \tau \in D\}$ lie in $H^{-1}([0, \lambda))$ when u does. \square

Proposition 3.17. *If (V, Ω) and (V', Ω') are two complete solutions with*

$$rad(V, \Omega) \leq rad(V', \Omega')$$

then letting $\lambda := rad(V, \Omega)$ we have that

$$H^{-1}([0, \lambda)) = (H')^{-1}([0, \lambda))$$

and $\Omega = \Omega'$ there.

Proof. Clearly we can assume that $\lambda > 0$ since otherwise the statement is vacuous. By Lemma 3.16 the solutions $(H^{-1}([0, \lambda)), \Omega)$ and $((H')^{-1}([0, \lambda)), \Omega')$ are still complete, so for ease of notation we may as well assume that $V = H^{-1}([0, \lambda))$ and $V' = (H')^{-1}([0, \lambda))$ (and hence $rad(V', \Omega') = \lambda$). Pick a point $u \in V_0$, then we know that $H(u) < \lambda$. We now use an argument similar to the one in the proof of Proposition 3.12. Thus we let \mathcal{L}_t denote the leaf in the Monge-Ampère foliation of Ω that passes through tu for $t \in [0, 1]$. We also let \mathcal{L}'_t denote the leaf in the Monge-Ampère foliation of Ω' that passes through tu , if there is any such leaf. We let $T := \sup\{t : \mathcal{L}_t = \mathcal{L}'_t\}$ and again we claim that $T = 1$. That the set of points $t \in [0, 1]$ such that $\mathcal{L}_t = \mathcal{L}'_t$ is open follows exactly as before. On the other hand,

$$\lim_{t \rightarrow T} H'(tu) = \lim_{t \rightarrow T} H(tu) < \lambda$$

which since $rad(V', \Omega') = \lambda$ implies that $Tu \in V'$. Then the closedness follows as before since the leaves vary continuously with t , which exactly as in the proof of Proposition 3.12 gives us the desired equalities. \square

Armed with this result it is immediate how to construct a canonical complete solution (V_{can}, Ω_{can}) when the canonical radius Λ_{can} is positive. Namely, let (V_i, Ω_i) be any sequence of complete solutions such that $\lambda_i := rad(V_i, \Omega_i)$ is increasing to Λ_{can} . Then we define

$$V_{can} := \cup_i H_i^{-1}([0, \lambda_i))$$

and we let Ω_{can} be defined to be equal to Ω_i on $H_i^{-1}([0, \lambda_i))$. That this is a well-defined complete solution now follows immediately from Proposition 3.17. We have thus proved Theorem 1.6 from the Introduction:

Theorem 3.18. *Let Y be compact (or more generally assume that $\Lambda_{can} > 0$). Then there is a unique complete solution (V_{can}, Ω_{can}) such that*

$$rad(V_{can}, \Omega_{can}) = \Lambda_{can}$$

and

$$H_{can} < \Lambda_{can}.$$

This canonical solution is maximal in the following sense. If (V, Ω) is any other complete solution then for any $\lambda < rad(V, \Omega)$ we have that

$$H^{-1}([0, \lambda)) = H_{can}^{-1}([0, \lambda))$$

and there $\Omega = \Omega_{can}$.

4. TUBULAR NEIGHBOURHOODS

Let Y be a submanifold in a Kähler manifold (X, ω) . Recall that $\pi: N_Y \rightarrow Y$ denotes the normal bundle and $\iota: Y \rightarrow N_Y$ is the inclusion of Y as the zero section in N_Y .

Definition 4.1. A *tubular neighbourhood* of Y in X is a smooth map $T: U \rightarrow X$ from an open neighbourhood U of $\iota(Y)$ in N_Y which is a diffeomorphism onto a neighbourhood $T(U)$ of Y with $T \circ \iota = \text{id}_Y$.

Suppose that Ω is as provided by Theorem 3.10, so is a regular solution to the HMAE on a neighbourhood $V \subset \mathcal{N}_Y$ of the proper transform \mathcal{Y} of $Y \times D$ with boundary data ω . Recall that $U := V_0$ denotes the central fibre of V , which is a neighbourhood of $\iota(Y) \subset N_Y$.

Definition 4.2. We let ω_{N_Y} be the restriction of the regular solution Ω of the HMAE to U .

So ω_{N_Y} is an S^1 -invariant Kähler form on U , and by the uniqueness property of regular solutions, the germ of this Kähler form around $\iota(Y) \subset N_Y$ is independent of choice of Ω . We recall that V was assumed to be a union of complete leaves of the Monge-Ampère foliation determined by Ω . Thus flowing along these leaves gives an injective smooth map

$$\hat{T}: U \times D \rightarrow V$$

such that $\pi_D \hat{T}(u, \tau) = \tau$ and for each $u \in U$ the map $\hat{T}_u(\tau) = \hat{T}(u, \tau)$ is holomorphic. The family of maps $\hat{T}(\cdot, \tau)$ has an additional remarkable property.

Proposition 4.3. *For any $\tau \in D$ we have that*

$$\hat{T}(\cdot, \tau)^*(\Omega|_{V_\tau}) = \omega_{N_Y}. \quad (7)$$

Proof. This is classical (see e.g. [3] or [12]) but for the convenience of the reader we give the simple argument here.

When differentiating the left hand side of (7) by a vector field v on the base one gets

$$\hat{T}(\cdot, \tau)^*(L_{\tilde{v}}\Omega|_{V_\tau})$$

where \tilde{v} is the unique lift of v to V parallel to the foliation, and at the same time by Cartan's formula

$$L_{\tilde{v}}\Omega = \iota_{\tilde{v}}d\Omega + d\iota_{\tilde{v}}\omega = 0$$

since Ω is closed and \tilde{v} lies in the kernel. Thus the left hand side of (7) is independent of τ , so letting τ be zero completes the proof. \square

Definition 4.4. We define $T: U \rightarrow X$ by

$$T(u) = \hat{T}(u, 1) \quad (8)$$

Lemma 4.5. *$T: U \rightarrow X$ is a tubular neighbourhood of Y and $T^*\omega = \omega_{N_Y}$. In particular $T^*\omega$ is Kähler and S^1 -invariant.*

Proof. By construction $T : U \rightarrow X$ is a tubular neighbourhood. The second statement follows from Proposition 4.3. \square

The next result completes the Proof of Theorem 1.1 in the Introduction.

Proposition 4.6. *Let $u \in U$ and $f_u : S^1 \rightarrow X$ be given by $f_u(e^{i\theta}) = T(e^{i\theta}u)$ for $e^{i\theta} \in S^1$. Then there exists a holomorphic $F_u : D \rightarrow X$ extending f_u such that*

$$F_u(0) = \pi(u) \text{ and } \left[DF_u|_0 \left(\frac{\partial}{\partial x} \right) \right] = u.$$

Proof. For $u \in U$ there is a unique holomorphic leaf \mathcal{L}_u of the Monge-Ampère foliation that passes through u and by definition of T it contains the point $(T(u), 1)$. Now let F_u be the lift from D to \mathcal{L} composed with the projection to X . It is then immediate that $F_u(1) = T(u)$, $F_u(0) = \pi(u)$ and $[DF_u|_0(\frac{\partial}{\partial x})] = u$. The S^1 -invariance of the foliation implies that

$$\mathcal{L}_u = e^{-i\theta} \cdot \mathcal{L}_{e^{i\theta}u},$$

and thus $(T(e^{i\theta}u), e^{i\theta}) \in \mathcal{L}_u$. Thus we get that F_u extends f_u . \square

When Y is compact (or more generally $\Lambda_{can} > 0$) we clearly get a canonical tubular neighbourhood $T_{can} : U_{can} \rightarrow X$ with the desired properties by using the canonical complete solution (V_{can}, Ω_{can}) .

We also have the following characterization of when the germ of T is holomorphic at a given point $\iota(p) \in \iota(Y)$.

Proposition 4.7. *The germ of T is holomorphic around a point $\iota(p) \in \iota(Y)$ if and only if there exists a neighbourhood U_p of p together with a holomorphic S^1 -action on U_p such that $\omega|_{U_p}$ is invariant, $Y \cap U$ is fixed pointwise and the induced action on $N_{Y \cap U}$ is equal to the usual rotation of its fibers.*

Proof. If the germ of T is holomorphic at $\iota(p)$ then we can take U_p to be the image under T of an S^1 -invariant neighbourhood of $\iota(p)$, and consider the induced action of S^1 on the image. This will then have the described properties since $T^*\omega$ is S^1 -invariant.

Let now U_p be a neighbourhood of p with the properties described above. Choose holomorphic coordinates z_i centered at p such that locally around p we have that Y is given by the equations $z_i = 0$, $1 \leq i \leq r$. Let σ denote the S^1 -action on U_p . For $1 \leq i \leq r$ we let \tilde{z}_i be the holomorphic function defined by

$$\tilde{z}_i(x) := \frac{1}{2\pi i} \int z_i(\sigma(\tau) \cdot x) \tau^{-2} d\tau,$$

while for $r < i \leq n$ we let

$$\tilde{z}_i(x) := \frac{1}{2\pi i} \int z_i(\sigma(\tau) \cdot x) \tau^{-1} d\tau.$$

Then these coordinates \tilde{z}_i define a ρ -chart $f^{-1} : U \rightarrow B_\delta$ and it is easy to see that on this chart $\rho = \sigma$. Thus by assumption ω_f is ρ -invariant, giving rise to the trivial local regular solution $\pi_{B_\delta}^* \omega_f$. The associated foliation is the trivial one, and by uniqueness (Proposition 3.7) the Monge-Ampère foliation of any regular solution agrees with this trivial one near D_p . In particular we see that T is holomorphic near $\iota(p)$. \square

The next definition captures another relation between T and the holomorphic structure.

Definition 4.8. Let U, X be complex manifolds with fixed inclusions $U \subset V$ and $X \subset V$ for some complex manifold V . We say that a smooth map $T : U \rightarrow X$ is *holomorphic motion* if there exists a smooth

$$\hat{T} : U \times D \rightarrow V$$

such that

- (1) $\hat{T}(u, 0) = u$ and $\hat{T}(u, 1) = T(u)$ for all $u \in U$.
- (2) For each $u \in U$ the map $w \mapsto \hat{T}(u, w)$ is holomorphic.
- (3) For each $w \in D$ the map $u \mapsto \hat{T}(u, w)$ is injective.

So by construction $T: U \rightarrow T(U)$ is a holomorphic motion (with respect to V).

Remark 4.9. This terminology is adapted from the case of a family of injections $f_\tau: E \rightarrow \mathbb{P}^1$ parameterized by $\tau \in D$ which for fixed $x \in E$ are holomorphic in τ ; we refer the interested reader to [13] and the references therein.

5. LOCAL REGULARITY OF WEAK SOLUTIONS

Recall that one cannot find global regular solutions of the HMAE on \mathcal{N}_Y but instead of looking for local regular solutions one can consider global weak solutions.

If Ψ is a $\pi_X^* \omega$ -psh function on \mathcal{N}_Y and $\nu_E(\Psi) \geq c$ then $\pi_X^* \omega + dd^c \Psi - c[E]$ is a closed positive current cohomologous to $\pi_X^* \omega - c[E]$. Pick a $c > 0$ and let Φ_w be defined as the supremum of all $\pi_X^* \omega$ -psh functions on \mathcal{N}_Y such that $\Psi \leq 0$ on $X \times S^1$ and with $\nu_E(\Psi) \geq c$.

We call the closed positive $(1, 1)$ -current on \mathcal{N}_Y :

$$\Omega_w := \pi_X^* \omega + dd^c \Phi_w - c[E]$$

the weak solution to the HMAE on \mathcal{N}_Y . A motivation for this comes from the following theorem.

Theorem 5.1. *When X is compact and c sufficiently small the closed positive $(1, 1)$ -current*

$$\Omega_w := \pi_X^* \omega + dd^c \Phi_w - c[E]$$

solves the weak HMAE on \mathcal{Y} with boundary condition induced by ω , i.e.

$$\Omega_w^{n+1} = 0, \tag{weak-HMAE}$$

$$\Omega|_{\pi^{-1}(\tau)} = \omega \text{ for } \tau \in S^1.$$

Proof. This is proved by Berman in [4, Prop 2.7] which in turn relies on the original approach of Bedford-Taylor [2] (see also Demailly [10, 12.3]). The cited reference is stated for certain kinds of “test configurations”, an example of which is \mathcal{N}_Y , and under some additional assumptions such as X being Fano, and $[\omega]$ being integral, but these are not used in the proof. We stress that since Φ_w is not necessarily smooth, the term Ω_w^{n+1} is to be taken in the sense of Bedford-Taylor. Also the boundary value statement in (weak-HMAE) should be read as saying that $\Phi_w(x, \tau) \rightarrow 0$ uniformly as $\tau \rightarrow 1$. \square

Theorem 5.2. *Assume that Y is compact (or more generally $\Lambda_{can} > 0$) and let (V_{can}, Ω_{can}) be the canonical complete solution as provided by Theorem 1.6. Then*

$$\Omega_w = \Omega_{can}$$

on $H_{can}^{-1}([0, c])$. In particular Ω_w is regular in a neighbourhood of \mathcal{Y} .

To prove Theorem 5.2 we will consider the Legendre transform of $\Phi := \Phi_{can}$, i.e. the unique S^1 -invariant function on $V := V_{can}$ being zero on V_1 such that $\Omega_{can} = \pi_X^* \omega + dd^c \Phi$.

So for $0 \leq \lambda < \Lambda_{can}$ we consider the Legendre transform

$$\alpha_\lambda(x) := \inf \{ \Phi(x, \tau) + \lambda \ln |\tau|^2 : (x, \tau) \in V \cap (X \times D^\times) \}. \tag{9}$$

Note that the function $g(t) := \Phi(x, e^{-t/2}) - \lambda t$ is convex and

$$g'(t) = H(x, e^{-t/2}) - \lambda, \tag{10}$$

where $H := H_{can}$ is the Hamiltonian. Since $H \rightarrow \Lambda_{can}$ as (x, τ) approaches ∂V it follows that the infimum in (9) is attained in V .

Proposition 5.3. *For $\lambda < \Lambda_{can}$ the function α_λ is ω -psh, $\{\alpha_\lambda < 0\} = H^{-1}([0, \lambda))$, and $\nu_Y(\alpha_\lambda) \geq \lambda$.*

Proof. That α_λ is ω -psh follows from the Kiselman minimum principle [20], [11, ChI 7B]. Also from (10) we see that $\alpha_\lambda < 0$ precisely on $H^{-1}([0, \lambda))$. For the final statement we will work locally around one of the discs \mathcal{D}_p . So let $f^{-1} : U \rightarrow B_\delta$ be an admissible ρ -chart centered at p . Then we can write $\omega_f = dd^c \phi$ where $\phi(z) = |z|^2 + O(|z|^3)$ and

$$\Phi' := \Phi \circ \Gamma + \phi \circ \rho.$$

We know that $\phi \leq C|z|^2$ on B_δ for some number $C > 1$ and it follows that $\Phi' \leq C|z|^2$ on $B_\delta \times D$. Since as we recall

$$\Gamma(e^{t/2}z', z'', e^{-t/2}) = (f(z', z''), e^{-t/2})$$

it follows that

$$\begin{aligned} \alpha_\lambda(f(z)) + \phi(z) &\leq \inf\{\Phi'(e^{t/2}z', z'', e^{-t/2}) - \lambda t : t \in [0, T]\} \leq \\ &\leq C(e^T|z'|^2 + |z''|^2) - \lambda T = C\delta^2 - \lambda \ln(\delta^2 - |z''|^2) + \lambda \ln |z'|^2. \end{aligned}$$

where

$$T := \ln \left(\frac{\delta^2 - |z''|^2}{|z'|^2} \right),$$

i.e. $(e^{T/2}z', z'') \in \partial B_\delta$. This shows that $\nu_Y(\alpha_\lambda) \geq \lambda$. \square

Proof of Theorem 5.2. In light of Proposition 5.3 we can define α_λ to be zero on $X \setminus V_1$ and thus get an ω -psh function on X with Lelong number along Y greater than or equal to λ .

Now assume that $\lambda < c$. Then the function $\alpha_\lambda(x) + (c - \lambda) \ln |\tau|^2$ is $\pi_X^* \omega$ -psh, bounded from above by zero on $X \times S^1$ and has Lelong number at least c along E . From the definition of Φ_w as the supremum of such functions we get that

$$\alpha_\lambda(x) + (c - \lambda) \ln |\tau|^2 \leq \Phi_w.$$

On the other hand, by the involution property of the Legendre transform

$$\Phi(x, \tau) + c \ln |\tau|^2 = \sup\{\alpha_\lambda(x) + (c - \lambda) \ln |\tau|^2 : 0 \leq \lambda < \min(c, \Lambda)\}$$

on $H^{-1}([0, c))$, and thus we get that

$$\Phi(x, \tau) + c \ln |\tau|^2 \leq \Phi_w$$

on $H^{-1}([0, c))$.

Pick a leaf in the foliation of $(H^{-1}([0, c)), \Omega_{can})$ and let ψ be a function on it such that $dd^c \psi = \pi_X^* \omega|_{\mathcal{L}}$. Since Φ_w has Lelong number c along E it follows that $(\Phi_w - c \ln |\tau|^2)|_{\mathcal{L}} + \psi$ defines a subharmonic function on \mathcal{L} , which is equal to ψ on the boundary. On the other hand we know that $\Phi|_{\mathcal{L}} + \psi$ is harmonic and equal to ψ on the boundary, which implies that in fact

$$\Phi_w = \Phi + c \ln |\tau|^2$$

on $H^{-1}([0, c))$. Locally on a ρ -chart we would get that

$$\Phi_w \circ \Gamma = \Phi \circ \Gamma + c \ln |\tau|^2$$

on $H^{-1}([0, c)) \cap B_\delta \times D$, and since in this picture E is the zero fiber this then implies that

$$\Omega_w = \Omega_{can}$$

on $H^{-1}([0, c))$. \square

6. OPTIMAL REGULARITY OF ENVELOPES

Recall that the envelope ψ_λ we wish to consider is

$$\psi_\lambda := \sup\{\psi \in PSH(X, \omega) : \psi \leq 0, \nu_Y(\psi) \geq \lambda\},$$

and the equilibrium set S_λ is

$$S_\lambda := \{\psi_\lambda = 0\}.$$

We also recall the definition of optimal regularity for such envelopes.

Definition 6.1. We say that ψ_λ has optimal regularity if S_λ is smoothly bounded, the function ψ_λ is smooth on $S_\lambda^c \setminus Y$,

$$(\omega + dd^c \psi_\lambda)^{n-1} \neq 0 \text{ on } S_\lambda^c \setminus Y,$$

and the blowup of S_λ^c along Y is foliated by holomorphic discs attached to ∂S_λ passing through Y such that the restriction of $\omega + dd^c \psi_\lambda$ to each such disc vanishes.

Theorem 6.2. *Suppose Y is compact (or more generally $\Lambda_{can} > 0$). Then for λ small enough (i.e. $\lambda < \Lambda_{can}$) the envelopes ψ_λ have optimal regularity, the boundaries ∂S_λ vary smoothly with λ , and the corresponding holomorphic discs attaching to ∂S_λ and passing through Y all have area λ .*

Proof. Let (V_{can}, Ω_{can}) be the canonical solution to the HMAE with boundary data induced by ω , as provided by Theorem 1.6, and choose a number c such that $\lambda < c \leq \Lambda_{can}$ (if Λ_{can} is finite we can just as well set $c = \Lambda_{can}$). Let Φ_w be corresponding weak solution as defined in Section 5. We saw in the proof of Theorem 5.2 that

$$\Phi_w = \Phi + c \ln |\tau|^2$$

in $H^{-1}([0, c])$ (here as we recall Φ is the unique S^1 -invariant function on \mathcal{N}_Y equal to zero on $X \times S^1$ such that $\Omega_{can} = \pi_X^* \omega + dd^c \Phi$ and H is the Hamiltonian of (V_{can}, Ω_{can})).

We get by definition that

$$\psi_\lambda + (c - \lambda) \ln |\tau|^2 \leq \Phi_w,$$

which implies that

$$\psi_\lambda(x) \leq \inf_{\tau \in D^\times} \{(\Phi_w(x, \tau) - c \ln |\tau|^2) + \lambda \ln |\tau|^2\}. \quad (11)$$

Since $\Phi_w = \Phi + c \ln |\tau|^2$ in $H^{-1}([0, c])$ we see that the right hand side of (11) is equal to α_λ on $H^{-1}([0, c])$ and since it is less than or equal to zero on X it must be equal to α_λ on the whole of X . Thus $\psi_\lambda \leq \alpha_\lambda$, but since α_λ is a candidate for the supremum we get that in fact

$$\psi_\lambda = \alpha_\lambda.$$

Thus from Proposition 5.3 we get that the boundary ∂S_λ is equal to $H^{-1}(\lambda) \cap V_1$. Since H is constant along the leaves of the foliation it follows that ∂S_λ is the image of the λ level set of H on V_0 under the tubular neighbourhood T . Since Ω restricted to V_0 is Kähler and ζ (i.e. the vector field generating the S^1 -action) is nonvanishing on $N_Y \setminus \iota(Y)$ we get that $dH = \iota_\zeta \Omega$ is nonzero away from $\iota(Y)$. This shows that ∂S_λ is smooth and varies smoothly with λ .

Let now x be a point in $S_\lambda^c \setminus Y$. Since $H^{-1}(\lambda)$ is foliated by leaves there is one such leaf \mathcal{L} which intersects the open line segment $\{(x, \tau) : \tau \in (0, 1)\}$. At the intersection point (x, τ) we then have that

$$\alpha_\lambda(x) = \Phi(x, \tau) + \lambda \ln |\tau|^2.$$

On \mathcal{L} , $dd^c \Phi = -\pi^* \omega$ and thus the function

$$\alpha_\lambda(x) - \Phi(x, \tau) - \lambda \ln |\tau|^2$$

restricted to \mathcal{L} is thus subharmonic, bounded from above by zero and equal to zero at an interior point (x, τ) , and thus must be equal to zero. Therefore

$$\alpha_\lambda(x) = \Phi(x, \tau) + \lambda \ln |\tau|^2 \quad (12)$$

on $H^{-1}(\lambda)$ and

$$dd^c \alpha_\lambda(x, \tau) = -\pi_X^* \omega \quad (13)$$

on \mathcal{L} . This now shows that $\alpha_\lambda = \psi_\lambda$ is smooth on $S_\lambda^c \setminus Y$. If we then project the leaf \mathcal{L} to X we get a holomorphic disc \mathcal{L}_x passing through Y which attaches to ∂S_λ along its boundary. Also because of (13) we have that $\omega + dd^c \psi_\lambda$ vanishes on \mathcal{L}_x . It is clear that these discs foliate the blowup of S_λ^c along Y . To calculate the area of one of these discs \mathcal{L}_x we first observe that

$$\int_{\mathcal{L}_x} \omega = \int_{T^{-1}(\mathcal{L}_x)} \omega_{N_Y}.$$

Now $T^{-1}(\mathcal{L}_x)$ is not necessarily a holomorphic disc in N_Y but the boundary we know is a circle $e^{i\theta}u$ and so the symplectic area is the same as for the holomorphic disc $\{\tau u : \tau \in D\}$. Since Ω_{N_Y} is S^1 -invariant and $H = \lambda$ on the boundary of this disc this means that it has symplectic area λ .

Finally we will show that $(\omega + dd^c \psi_\lambda)^{n-1} \neq 0$ on $S_\lambda^c \setminus Y$. Pick a complex hyperplane in the tangent space at a point $x \in S_\lambda^c \setminus Y$ not containing the tangents of the holomorphic disc going through x . This is then the projection of a complex subspace of the tangent space of \mathcal{N}_Y at (x, τ) , and we can choose this subspace to lie in the tangent space of $H^{-1}(\lambda)$. This subspace will not contain the tangents of the leaf going through (x, τ) , and thus Ω_{can}^{n-1} is a volume form on it. Because of (12) this implies that $(\omega + dd^c \psi_\lambda)^{n-1} \neq 0$. \square

Remark 6.3. In fact the proof shows that the pair $(S_\lambda^c \setminus Y, \omega + dd^c \psi_\lambda)$ is canonically identified with the (semi)symplectic quotient $H^{-1}(\lambda)/S^1$ of $(V_{can} \setminus \mathcal{Y}, \Omega_{can})$.

When Y is just a point Theorem 1.9 reduces to the following local statement.

Theorem 6.4. *Let ϕ be a smooth strictly plurisubharmonic function on the unit ball B in \mathbb{C}^n . For small λ (i.e. $\lambda < \Lambda_{can}$) the envelope*

$$\psi_\lambda := \sup\{\psi \in PSH(B) : \psi \leq \phi - \lambda \ln |z|^2\}$$

has optimal regularity (here $S_\lambda := \{\psi_\lambda + \lambda \ln |z|^2 = \phi\}$ and the holomorphic discs will pass through the origin). Also, for any bounded holomorphic function f on

$$B_\lambda := S_\lambda^c$$

we have that

$$\frac{1}{\text{vol}(B_\lambda)} \int_{B_\lambda} f dV_\phi = f(0),$$

where the dV_ϕ denotes the Monge-Ampere measure $(dd^c \phi)^n/n!$

Proof. We get from the envelope ψ_λ to the kind considered earlier by taking $\psi_\lambda - \phi + \lambda \ln |z|^2$. It then follows from Theorem 6.2 that ψ_λ has optimal regularity as long as λ is small.

Now let f be a bounded holomorphic function on B_λ . Also let $T : U \rightarrow B$ be the canonical tubular neighbourhood of 0 in $(B, dd^c \phi)$, thus U is an S^1 -invariant neighbourhood of the origin in $T_0 B \cong \mathbb{C}^n$, and $\tilde{\omega} := T^* dd^c \phi$ is Kähler and invariant. We also note that

$T^{-1}(B_\lambda) = H^{-1}([0, \lambda)) \cap U$ and so this is also invariant. Now pulling back the integral we get that

$$\begin{aligned} \int_{B_\lambda} f dV_\phi &= \int_{\{H \leq \lambda\}} f(T(z)) \tilde{\omega}^n = \int_{\tau \in S^1} \left(\int_{\{H \leq \lambda\}} f(T(\tau z)) \tilde{\omega}^n \right) d\tau = \\ &= \int_{\{H \leq \lambda\}} \left(\int_{\tau \in S^1} f(T(\tau z)) d\tau \right) \tilde{\omega}^n. \end{aligned}$$

Since the map $\tau \mapsto T(\tau z)$ extends to a holomorphic disc in B passing through the origin at its centre it follows from the mean value property that

$$\int_{\tau \in S^1} f(T(\tau z)) d\tau = f(0),$$

at which point we are done. \square

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